Comment on "Self-interacting Elko dark matter with an axis of locality"

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In this comment we show that the statement in Ahluwalia *et al.* [Phys. Rev. D **83**, 065017 (2011)] that the existence of Elko spinor fields imply in an *axis of locality* is equivocated. The anticommutator $\{\Lambda(\mathbf{x},t),\Pi(\mathbf{x},t)\}$ is strictly local.

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I. THE INTEGRAL APPEARING IN THE PROPAGATOR

In Ref. [1] the authors calculated the propagator for an Elko spinor field supposed to satisfy the Klein-Gordon equation in Minkowski spacetime. They obtained a sum of two terms, the first being the usual propagator (fundamental solution) for a Klein-Gordon field and the second involves evaluation of the integral [see their Eq. (6.20)]

$$\int \frac{d^4p}{(2\pi)^4} e^{-ip_\mu(x'^\mu - x^\mu)} \frac{i\varpi}{p_\mu p^\mu - m^2 + i\varepsilon} \mathcal{G}(\mathbf{p}). \qquad (1)$$

The calculation is done in an inertial reference frame[1] $\mathbf{e}_0 = \partial/\partial t$ with arbitrary spatial axes $\langle \mathbf{e}_1 = \frac{\partial}{\partial x}, \mathbf{e}_2 = \frac{\partial}{\partial y}, \mathbf{e}_3 = \frac{\partial}{\partial z}\rangle$ chosen in such a way that together with $\mathbf{e}_0$ defines a global orthonormal tetrad in Minkoswki spacetime. We next introduce spherical coordinates associated with the selected orthonormal triad $\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\rangle$ and write

$$\mathbf{p} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta).$$

Then[2]

$$\mathcal{G}(\mathbf{p}) := \gamma^5\gamma^\mu n_\mu, \qquad (2)$$

where the spacelike vector field $n = n^\mu \mathbf{e}_\mu$ is

$$n_\mu := (0, \mathbf{n}),$$
$$\mathbf{n} := \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{\mathbf{p}}{|\mathbf{p}|}\right) = (-\sin\varphi, \cos\varphi, 0).$$

Then the authors claim:

If there is no preferred direction, and since we are integrating over all momenta, we are free to choose a coordinate system in which $\mathbf{x}' - \mathbf{x}$ lies in the $\hat{z}$ direction. In this special case, the $\mathbf{p}\cdot(\mathbf{x}' - \mathbf{x})$ depends only on $p(=|\mathbf{p}|)$ and θ, but not on φ. Thus, the only φ-dependence in the whole integrand comes from $\mathcal{G}$ which depends on φ in such a manner that an integral over one period vanishes.

Remark I.—It is very important to remark that if this integral would result as non-null, the fundamental solution for the Klein-Gordon operator would have an additional term, something that obviously cannot be the case.

II. THE INTEGRAL OF $\mathcal{G}(\mathbf{p})$

On the other hand in Ref. [4] authors calculated the anticommutator of an Elko spinor field with its canonical momentum getting their Eq. (42), i.e.,

$$\{\Lambda(\mathbf{x},t), \Pi(\mathbf{x},t\} = i\delta(\mathbf{x}-\mathbf{x}')\mathbb{I} + i\int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\mathcal{G}(\mathbf{p}). \qquad (3)$$

There they claim:

"Since the integral on the right hand side of Eq. (42) vanishes only along the $\pm\hat{z}_e$ axis, the preferred axis also becomes the axis of locality."

Let us examine if that claim is correct. Call $|\mathbf{x}-\mathbf{x}'| = \Delta$, and put

$$(\mathbf{x}-\mathbf{x}') = \Delta(\sin\theta_\Delta\cos\varphi_\Delta, \sin\theta_\Delta\sin\varphi_\Delta, \cos\theta_\Delta). \qquad (4)$$

Calculation of the integral in the second member of Eq. (3) resumes in the calculation of the following integrals[3]

$$\mathbf{J}(\Delta) = \int_0^\infty dr r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi e^{ir\Delta f(\theta,\theta_\Delta,\varphi,\varphi_\Delta)}\sin\varphi, \qquad (5)$$

and

$$\mathbf{K}(\Delta) = \int_0^\infty dr r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi e^{ir\Delta f(\theta,\theta_\Delta,\varphi,\varphi_\Delta)}\cos\varphi, \qquad (6)$$

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[1]In relativity theory reference frames are represented by timelike vector fields $\mathbf{Z}$ on the manifold modeling spacetime. In special relativity if D is the Levi-Civita connection of the Minkowski metric η, an inertial frame is a timelike vector field $\mathbf{I}$ such that $D\mathbf{I} = 0$. Details can be found, e.g., in Refs. [2,3].

[2]See also Eq. (31) in Ref. [4].

[3]We take $\Delta \neq 0$. For $\Delta = 0$ it is obvious that $\mathbf{J}(0) = \mathbf{K}(0) = 0$.

with $f(\theta,\theta_\Delta,\varphi,\varphi_\Delta)=\sin\theta\cos\varphi\sin\theta_\Delta\cos\varphi_\Delta+\sin\theta\sin\varphi\times \sin\theta_\Delta\sin\varphi_\Delta+\cos\theta\cos\theta_\Delta$.

We will now calculate the integrals in Eqs. (5) and (6) in the cases when $\mathbf{x}-\mathbf{x}'$ lies, respectively, in the $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ directions. We will call the respective integrals $\mathbf{J}_i(\Delta)$ and $\mathbf{K}_i(\Delta)$, for $i=1, 2, 3$.

We start with the observation that it is trivial to verify that $\mathbf{J}_3(\Delta)=0$ and $\mathbf{K}_3(\Delta)=0$. To continue we calculate $\mathbf{J}_2(\Delta)$.

So, let us choose the spatial axis such that $(\mathbf{x}-\mathbf{x}')=\Delta\mathbf{e}_2=\Delta(0,1,0)$ and perform the nontrivial exercise of calculating the value of the integral given by Eq. (5) in this case, i.e.,

$$\mathbf{J}_2(\Delta)=\int_0^\infty drr^2\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\varphi\sin\varphi e^{ir\Delta\sin\theta\sin\varphi}. \tag{7}$$

We start evaluating the φ-integral,

$$\mathbf{\Omega}(\xi)=\int_0^{2\pi}d\varphi\sin\varphi e^{i\xi\sin\varphi}, \tag{8}$$

where $\xi:=r\Delta\sin\theta$. Observe that $\Omega(\xi)=-id\Lambda(\xi)/d\xi$ where (see 8.473-4, page 968 of Ref. [5])

$$\begin{aligned}\Lambda(\xi)&=\int_0^{2\pi}d\varphi e^{i\xi\sin\varphi}\\&=2\int_0^\pi d\varphi\cos(r\Delta\sin\theta\sin\varphi)\\&=2\pi J_0(r\Delta\sin\theta),\end{aligned} \tag{9}$$

with J_0 the zero order Bessel function. So,

$$\Omega(\xi)=2\pi iJ_1(r\Delta\sin\theta), \tag{10}$$

where J_1 is the first-order Bessel function. Next we evaluate

$$\mathbf{\Xi}(r,\Delta)=2\pi i\int_0^\pi d\theta\sin\theta J_1(r\Delta\sin\theta). \tag{11}$$

Using the relation 6.681-8, page 739 of Ref. [5], namely

$$\int_0^\pi dx\sin(2\mu x)J_{2\nu}(2a\sin x)=\pi\sin(\mu\pi)J_{\nu-\mu}(a)J_{\nu+\mu}(a), \tag{12}$$

valid for $\mathrm{Re}(\nu)>-1$ we see that identifying $2\mu=1$, $2\nu=1$, and $2a=r\Delta$ we can write

$$\mathbf{\Xi}(r,\Delta)=2\pi^2 iJ_0\Big(\frac{r\Delta}{2}\Big)J_1\Big(\frac{r\Delta}{2}\Big). \tag{13}$$

So, putting $t=r\Delta/2$ we have

$$\mathbf{J}_2(\Delta)=\frac{16\pi^2 i}{\Delta^3}\int_0^\infty dtt^2J_0(t)J_1(t). \tag{14}$$

Now, recall relation 6.626-3, page 715 of Ref. [5] (with $\beta=1$), namely

$$\int_0^\infty dxe^{-2\alpha x}xJ_0(x)J_1(x)=\frac{1}{2\pi}\left[\frac{K\Big(\frac{1}{\sqrt{1+\alpha^2}}\Big)-E\Big(\frac{1}{\sqrt{1+\alpha^2}}\Big)}{\sqrt{1+\alpha^2}}\right]. \tag{15}$$

Then we see that

$$\mathbf{J}_2(\Delta)=-\frac{4\pi i}{\Delta^3}\lim_{\alpha\to 0}\frac{d}{d\alpha}\left[\frac{K\Big(\frac{1}{\sqrt{1+\alpha^2}}\Big)-E\Big(\frac{1}{\sqrt{1+\alpha^2}}\Big)}{\sqrt{1+\alpha^2}}\right]. \tag{16}$$

Recalling relations 8.113 and 8.114, page 905 of Ref. [5], we have for the elliptic functions K and E

$$\begin{aligned}K\Big(\frac{1}{\sqrt{1+\alpha^2}}\Big)&=\frac{\pi}{2}{}_1F_2\Big(\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2}\Big),\\E\Big(\frac{1}{\sqrt{1+\alpha^2}}\Big)&=\frac{\pi}{2}{}_1F_2\Big(-\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2}\Big),\end{aligned} \tag{17}$$

where ${}_2F_1$ are Gauss hypergeometric functions. So,

$$\begin{aligned}&\int_0^\infty dxe^{-2\alpha x}xJ_0(x)J_1(x)\\&\quad=\frac{1}{4}\left[\frac{{}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2}\Big)-{}_2F_1\Big(-\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2}\Big)}{\sqrt{1+\alpha^2}}\right]\end{aligned} \tag{18}$$

and to end our long calculation we must evaluate the limit when $a\to 0$ of

$$\frac{d}{d\alpha}\left[\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2})-{}_2F_1(-\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2})}{\sqrt{1+\alpha^2}}\right]. \tag{19}$$

Recalling that (see, e.g., page 281 of Ref. [6])

$$\frac{d}{dz}{}_2F_1(a,b;c;z)=\frac{ab}{c}{}_2F_1(a+1,b+1;c+1;z),$$

and that ${}_2F_1(a,b;c;z)=\sum_{n=0}^\infty\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$ (with $(a)_n$, $(b)_n$, $(c)_n$ the Pochhammer symbols) we see that we are justified in writing

$$\begin{aligned}&\lim_{\alpha\to 0}\frac{d}{d\alpha}\left\{\frac{{}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2}\Big)}{\sqrt{1+\alpha^2}}\right\}\\&\quad=\lim_{\alpha\to 0}\left\{\begin{aligned}&-\alpha(1+\alpha^2)^{-3/2}{}_2F_1\Big(\tfrac{1}{2},\tfrac{1}{2};1;\tfrac{1}{1+\alpha^2}\Big)\\&-2\alpha(1+\alpha^2)^{-5/2}{}_2F_1'\Big(\tfrac{1}{2},\tfrac{1}{2};1;\tfrac{1}{1+\alpha^2}\Big)\end{aligned}\right\}=0.\end{aligned} \tag{20}$$

Also,

$$\lim_{\alpha\to 0}\frac{d}{d\alpha}\left[(1+\alpha^2)^{-1/2}{}_2F_1\Big(-\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2}\Big)\right]=0. \tag{21}$$

Finally, using Eqs. (20) and (21) in Eq. (16) we have that $\mathbf{J}_2(\Delta)=0$.

We now evaluate

$$\mathbf{K}_2(\Delta) = \int_0^\infty drr^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi e^{ir\Delta \sin\theta \sin\varphi} \cos\varphi. \tag{22}$$

Calling $\mathbf{x} = \sin\varphi$ and $r\Delta \sin\theta = \alpha$ we see that (taking into account relation 3.715-9, page 401 of Ref. [5] with $n = 1$) we have

$$\begin{aligned}\int_0^{2\pi} d\varphi e^{ir\Delta \sin\theta \sin\varphi} \cos\varphi &= \int_{-\pi}^{\pi} d\varphi e^{ir\Delta \sin\theta \sin\varphi} \cos\varphi \\ &= -2\int_0^\pi d\mathbf{x} \cos\varphi \cos(\mathbf{x} \sin\varphi) \\ &= -2[1 + (-1)]\frac{\pi}{2} J_1(\mathbf{x}) = 0\end{aligned} \tag{23}$$

and we conclude that $\mathbf{K}_2(\Delta) = 0$.

We now evaluate $\mathbf{J}_1(\Delta)$ and $\mathbf{K}_1(\Delta)$. Observe that

$$\int_0^{2\pi} d\varphi \sin\varphi e^{ir\Delta \sin\theta\cos\varphi} = -\int_{-\pi}^{\pi} d\varphi \sin\varphi e^{ir\Delta \sin\theta\cos\varphi} = 0, \tag{24}$$

since the integrand is an odd function and the interval is symmetric. So, we have

$$\mathbf{J}_1(\Delta) = \int_0^\infty drr^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi \sin\varphi e^{ir\Delta \sin\theta \cos\varphi} = 0. \tag{25}$$

It remains to evaluate

$$\mathbf{K}_1(\Delta) = \int_0^\infty drr^2 \int_0^\pi d\theta \sin\theta\left[\int_0^{2\pi} d\varphi e^{ir\Delta \sin\theta \cos\varphi} \cos\varphi\right]. \tag{26}$$

Writing $x = ir\Delta \sin\theta$ and changing $\varphi \mapsto \varphi + \pi$ the integral in the brackets in Eq. (26) becomes $2\pi i J_1(r\Delta \sin\theta)$. When this is inserted in Eq. (26) and steps analogous to the ones used in the evaluation of $\mathbf{J}_2(\Delta)$ are used, we get $\mathbf{K}_1(\Delta) = 0$.

III. CONCLUSIONS

We have shown through explicitly and detailed calculation that the integral of $\mathcal{G}(\mathbf{p})$ appearing in Eq. (42) of Ref. [4] is null for $\mathbf{x} - \mathbf{x}'$ lying in three orthonormal spatial directions in the rest frame of an arbitrary inertial frame $\mathbf{e}_0 = \partial/\partial t$.

This shows that no breakdown of locality concerning the anticommutator of $\{\Lambda(\mathbf{x}, t), \Pi(\mathbf{x}', t)\}$ and no preferred spacelike direction field in Minkowski spacetime is implied by the existence of Elko spinor fields.

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